

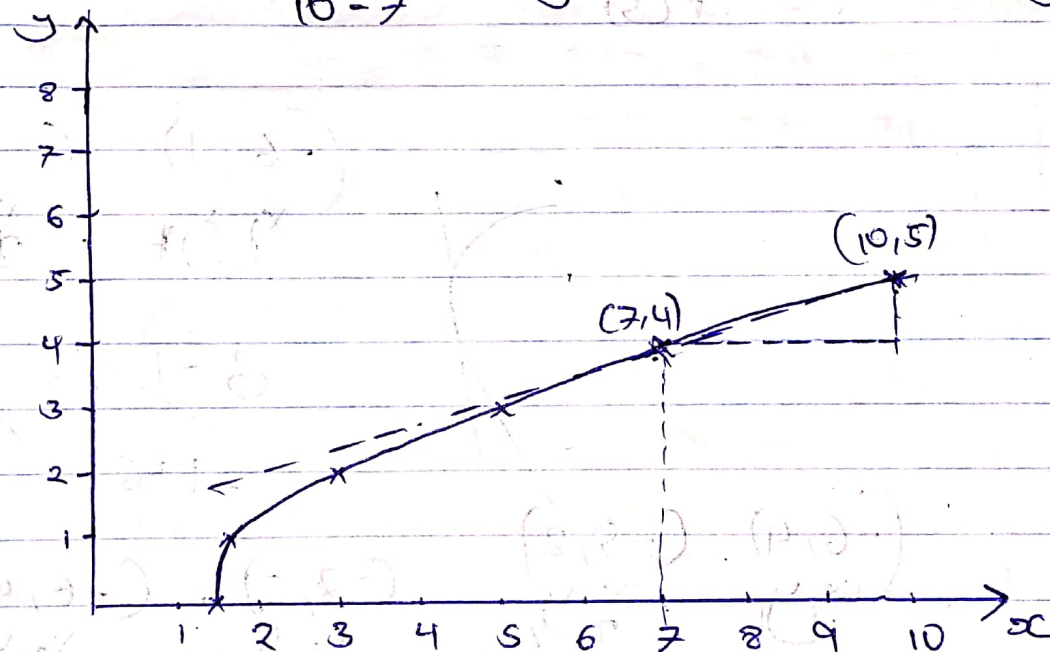
Practice Final Exam.

1
a) Estimating the value of $f'(7)$.

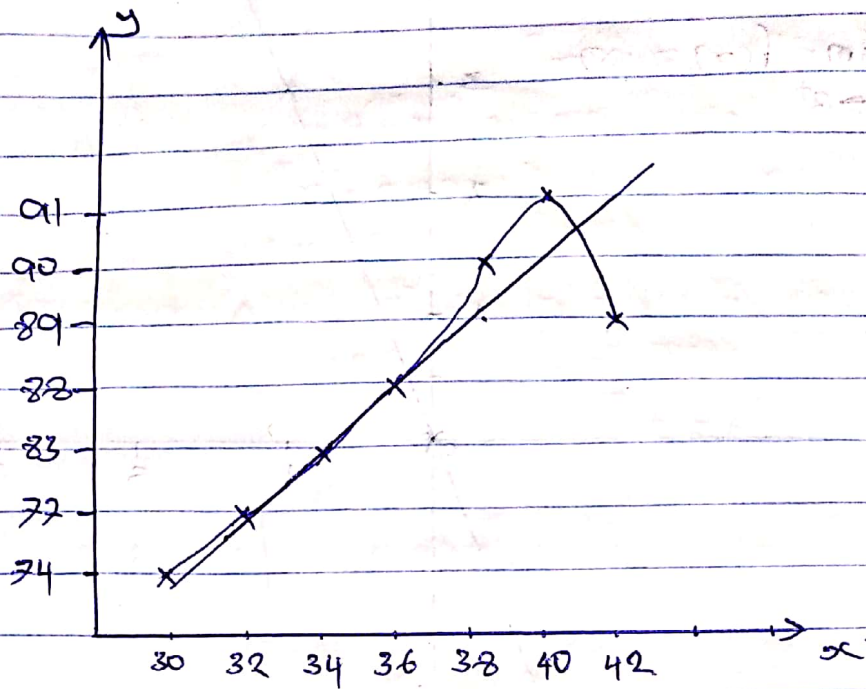
$$\begin{array}{cc} (7, 4) & (10, 5) \\ x_1, y_1 & x_2, y_2 \end{array}$$

$$= \frac{5-4}{10-7} = \frac{1}{3} \quad \text{thus } f'(7) = \frac{1}{3}$$

b).



2.



$$(36, 88) \quad (34, 83)$$

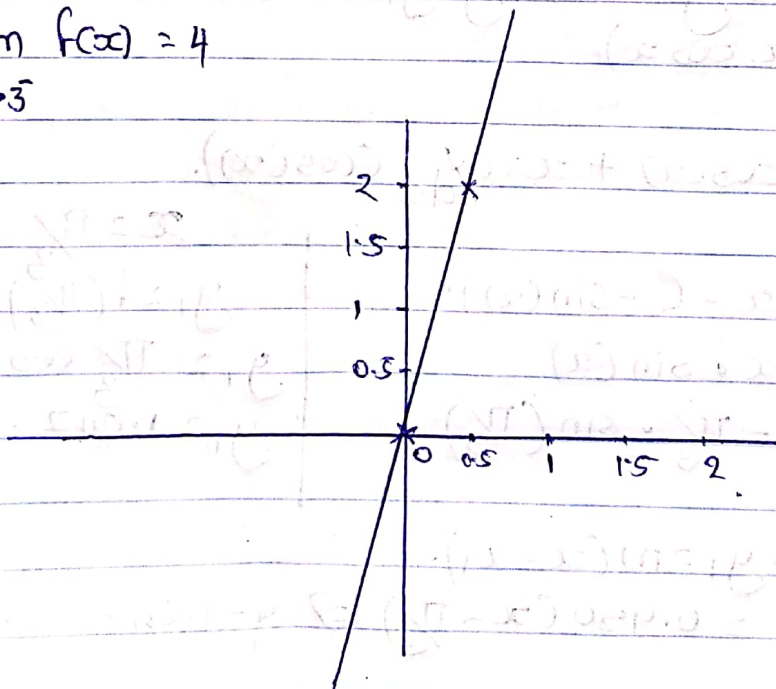
$$x_1, y_1 \quad x_2, y_2$$

$$= \frac{83 - 88}{34 - 36} = \frac{5}{2}$$

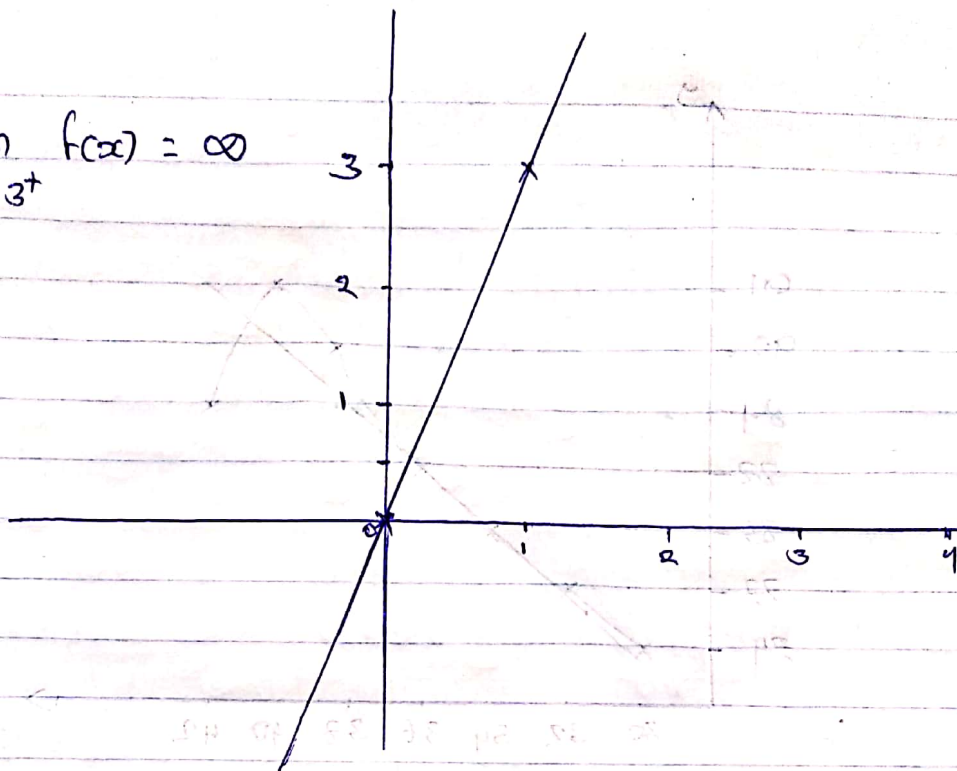
$$\text{Thus } f'(36) = \frac{5}{2}$$

3

$$a) \lim_{x \rightarrow 3^-} f(x) = 4$$



b) $\lim_{x \rightarrow 3^+} f(x) = \infty$



4

$$f(x) = x^3$$

$$f'(x) = \frac{d}{dx} (x^3)$$

$$f'(x) = 3x^2$$

$$f'(5) \Rightarrow 3(5)^2 = 75$$

5 $y = x \cos x$ at $x = \pi/3$

equation of tangent $\Rightarrow y - y_1 = m(x - x_1)$

$$= \frac{dy}{dx} (x \cos x)$$

$$= \frac{d}{dx} (x) \cdot \cos(x) + x \cdot \frac{d}{dx} (\cos(x))$$

$$= 1 \cos(x) + x \cdot (-\sin(x))$$

$$= \cos(x) - x \sin(x)$$

$$= \cos(\pi/3) - \pi/3 \cdot \sin(\pi/3)$$

$$m = 0.980$$

$$y - y_1 = m(x - x_1)$$

$$y - 1.047 = 0.980(x - \pi/3) \Rightarrow y - 1.047 = 0.980x - \frac{49\pi}{150}$$

$$\Rightarrow y = 0.980x + 0.02$$

$$\begin{aligned} x &= \pi/3 \\ y_1 &= f(\pi/3) \\ y_1 &= \pi/3 \cos \pi/3 \\ y_1 &= 1.047 \end{aligned}$$

$$6. \begin{cases} ax^2 + 1 & x < 2 \\ -x^2 + 6x + b & x \geq 2 \end{cases}$$

f is differentiable at $x = 2$.

so if f is continuous $x = 2 \Rightarrow \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$

$$\begin{array}{l|l} ax^2 + 1 & -x^2 + 6x + b \\ a(2)^2 + 1 & -(2)^2 + 6(2) + b \\ = 5a & = 8b \end{array}$$

$$a = 2$$

$$5a = 8b$$

$$5(2) = 8b$$

$$\frac{10}{8} = \frac{8b}{8}$$

$$b = \frac{5}{4}$$

thus $a = 2, b = \frac{5}{4}$

7 Calculate $f'(x)$

$$a) f(x) = \sqrt{3x-5}$$

$$= f'(x) = \frac{d}{dx} (\sqrt{3x-5})$$

$$= f'(x) = \frac{d}{dg} (\sqrt{g}) \times \frac{d}{dx} (3x-5)$$

$$g = 3x-5$$

$$= f'(x) = \frac{1}{2\sqrt{g}} \times 3$$

$$= f'(x) = \frac{1}{2\sqrt{3x-5}} \times 3$$

$$= f'(x) = \frac{3}{2\sqrt{3x-5}}$$

$$b) f(x) = \sin^{-1} x^3$$

$$= f(x) = \arcsin(x^3)$$

$$f'(x) = \frac{d}{dx} (\arcsin(x^3))$$

$$\text{using } \frac{d}{dx} (f(g)) = \frac{d}{dg} (f(g)) \times \frac{d}{dx} (g)$$

$$\text{where } g = x^3$$

$$= f'(x) = \frac{d}{dg} (\arcsin(g)) \times \frac{d}{dx} (x^3)$$

$$= f'(x) = \frac{1}{\sqrt{1-g^2}} \times 3x^2$$

$$= f'(x) = \frac{1}{\sqrt{1-(x^3)^2}} \times 3x^2$$

$$\Rightarrow f'(x) = \frac{3x^2}{\sqrt{1-x^6}}$$

8. Calculate $\frac{dy}{dx}$.

a) $y = e^{2x} \cos 3x.$

$$\frac{dy}{dx} = y' = \frac{d}{dx} (e^{2x} \cdot \cos(3x)).$$

$$\text{using } \frac{d}{dx} (f \times g) = \frac{d}{dx} (f) \times g + f \times \frac{d}{dx} (g).$$

$$\Rightarrow y' = \frac{d}{dx} (e^{2x}) \times \cos(3x) + e^{2x} \times \frac{d}{dx} (\cos(3x)).$$

$$\Rightarrow y' = e^{2x} \times 2 \cos(3x) + e^{2x} \times (-\sin(3x) \times 3)$$

$$= \frac{dy}{dx} = 2e^{2x} \times \cos(3x) - 3e^{2x} \times \sin(3x)$$

b) $y = \frac{\ln x}{\tan x}.$

$$= y' = \frac{d}{dx} \left(\frac{\ln(x)}{\tan(x)} \right)$$

$$\text{using } \frac{d}{dx} \left(\frac{f}{g} \right) = \frac{\frac{d}{dx} (f) \times g - f \times \frac{d}{dx} (g)}{g^2}.$$

$$\Rightarrow y' = \frac{\frac{d}{dx} (\ln(x)) \times \tan(x) - \ln(x) \times \frac{d}{dx} (\tan(x))}{\tan(x)^2}.$$

$$\Rightarrow y' = \frac{\frac{1}{x} \times \tan(x) - \ln(x) \sec(x)}{\tan(x)^2}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos(x) \sin(x) - x \cdot \ln(x)}{x \sin(x)^2}.$$

Q Calculate y'
 $\cos(xy) + y^2 - \ln x = 1$

$$\Rightarrow \cos(x) \cos(y) + y^2 - \ln(x) = 1$$

$$\Rightarrow \frac{d}{dx} (\cos(x) \cos(y)) + \frac{d}{dx} (y^2) - \frac{d}{dx} (\ln(x)) = \frac{d}{dx} (1)$$

$$\Rightarrow \frac{d}{dx} (\cos(x)) \times \cos(y) + \cos(x) \times \frac{d}{dx} (\cos(y)) + \frac{d}{dy} (y^2) \times \frac{dy}{dx} - \frac{1}{x} = 0$$

$$\Rightarrow -\sin(x) \cos(y) + \cos(x) \times (-\sin(y) \times \frac{dy}{dx}) + 2y \times \frac{dy}{dx} - \frac{1}{x} = 0$$

$$\Rightarrow -\sin(x) \cos(y) - \cos(x) \sin(y) \times \frac{dy}{dx} + 2y \times \frac{dy}{dx} - \frac{1}{x} = 0$$

$$\Rightarrow -\sin(x) \cos(y) - \cos(x) \sin(y) \times \frac{dy}{dx} + 2y \times \frac{dy}{dx} - \frac{1}{x} = 0$$

$$\Rightarrow -\sin(x) \cos(y) - \cos(x) \sin(y) \times \frac{dy}{dx} + 2y \times \frac{dy}{dx} - \frac{1}{x} = 0$$

$$\Rightarrow -\cos(x) \sin(y) \times \frac{dy}{dx} + 2y \times \frac{dy}{dx} = \sin(x) \cos(y) + \frac{1}{x}$$

$$\Rightarrow (-\cos(x) \sin(y) + 2y) \times \frac{dy}{dx} = \sin(x) \cos(y) + \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin(x) \cos(y) + \frac{1}{x}}{-\cos(x) \sin(y) + 2y}$$

$$\Rightarrow y' = \frac{x \cdot \sin(x) \cos(y) + 1}{-x \cdot \cos(x) \sin(y) + 2xy}$$

10

$$a) y = \log_3(x^2+1).$$

$$\Rightarrow y' = \frac{d}{dx} (\log_3(x^2+1))$$

using chain rule where $g = x^2+1$

$$\Rightarrow y' = \frac{d}{dg} (\log_3(g)) \times \frac{d}{dx} (x^2+1)$$

$$\Rightarrow y' = \frac{1}{\ln(3)g} \times 2x$$

$$\Rightarrow y' = \frac{1}{\ln(3) \times (x^2+1)} \times 2x$$

$$\Rightarrow y' = \frac{2x}{\ln(3) \times (x^2+1)}$$

$$b) y = 5^x$$

$$\Rightarrow y' = \frac{d}{dx} (5^x)$$

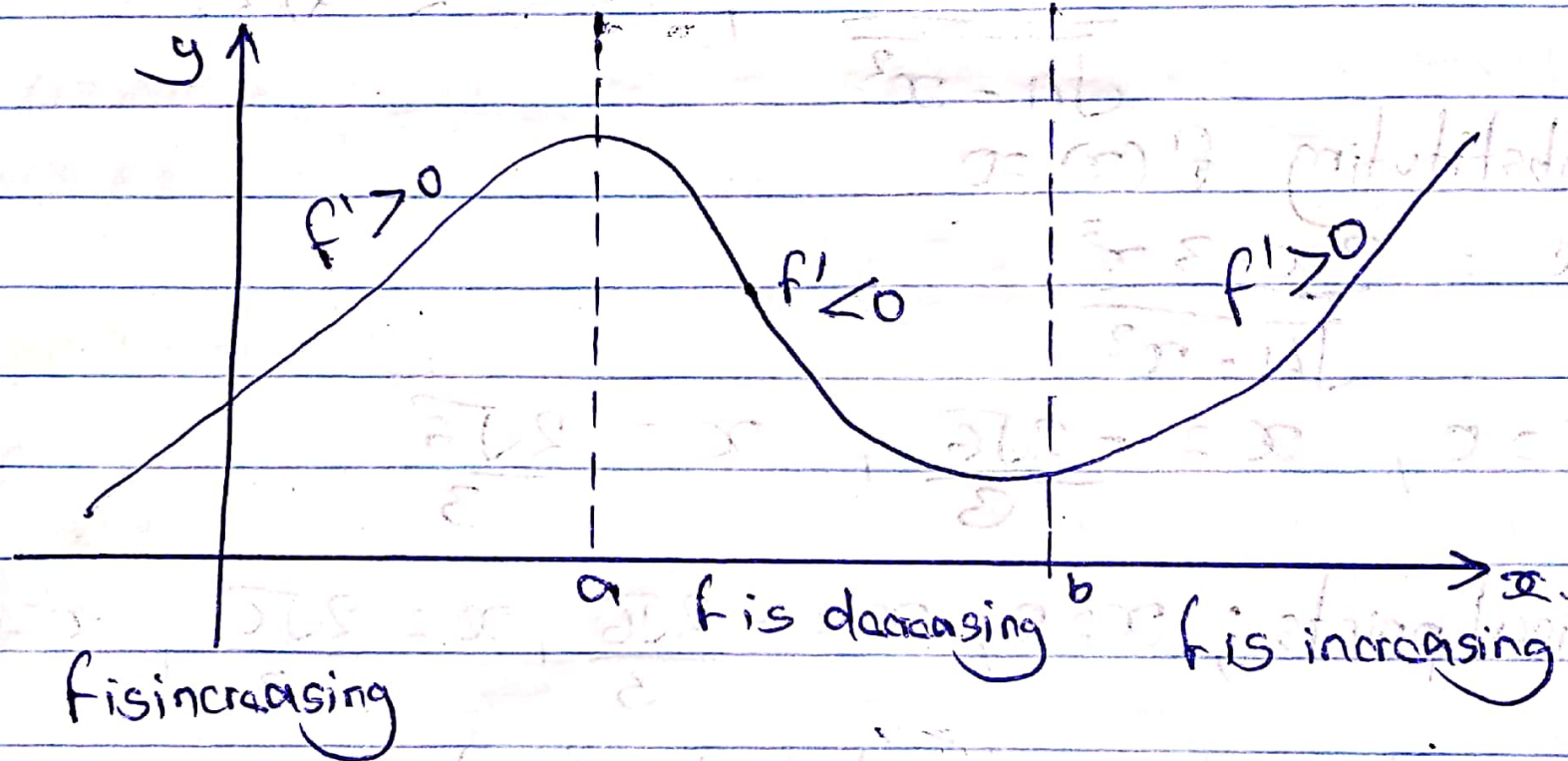
using $\frac{d}{dx} (a^x) = \ln(a) \times a^x$

$$\Rightarrow y' = \ln(5) \times 5^x$$

11

a) Rolle's theorem states that if a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) such that $f(a) = f(b)$, then $f'(c) = 0$ for some c with $a < c < b$.

b)



$$12. f(x) = x^2 \sqrt{4-x^2}$$

$$\Rightarrow f(x) = x^2 \sqrt{4-x^2}, \quad x \in [-2, 2]$$

$$\Rightarrow f'(x) = \frac{8x - 3x^3}{\sqrt{4-x^2}}$$

$$\Rightarrow f'(x) = \frac{8x - 3x^3}{\sqrt{4-x^2}}, \quad x \in (-2, 2)$$

substituting $f'(x) = 0$

$$0 = \frac{8x - 3x^3}{\sqrt{4-x^2}}$$

$$x = 0, \quad x = -\frac{2\sqrt{6}}{3}, \quad x = \frac{2\sqrt{6}}{3}$$

critical points: $x = 0, x = -\frac{2\sqrt{6}}{3}, x = \frac{2\sqrt{6}}{3}, x = -2, x = 2$

determination of the intervals, the function has no relative extrema

$$\left\langle -\frac{2\sqrt{6}}{3}, 0 \right\rangle, \left\langle 0, \frac{2\sqrt{6}}{3} \right\rangle$$

$$\left\langle -2, -\frac{2\sqrt{6}}{3} \right\rangle, \left\langle -\frac{2\sqrt{6}}{3}, 0 \right\rangle$$

$$\left\langle 0, \frac{2\sqrt{6}}{3} \right\rangle, \left\langle \frac{2\sqrt{6}}{3}, 2 \right\rangle$$

no relative extrema at $x = -2$

no relative extrema at $x = 2$

the points include:

$$x_1 = -1$$

$$x_4 = -1$$

$$x_2 = 1$$

$$x_5 = 1$$

$$x_3 = -\frac{19}{10}$$

$$x_6 = \frac{19}{10}$$

$$f'(-1) \approx -2.88675$$

$$f'(-1) = -2.88675$$

$$f'(1) \approx 2.88675$$

$$f'(1) = 2.88675$$

$$f'\left(-\frac{19}{10}\right) \approx 8.61009$$

$$f'\left(\frac{19}{10}\right) = -8.61009$$

The relative minimum is at $x=0$

The relative maximum is at $x = -\frac{2\sqrt{6}}{3}$

The relative maximum is at $x = \frac{2\sqrt{6}}{3}$

$$f(x) = x^2 \sqrt{4-x^2}, \quad x=0$$

$$f(x) = x^2 \sqrt{4-x^2}, \quad x = -\frac{2\sqrt{6}}{3}$$

$$f(x) = x^2 \sqrt{4-x^2}, \quad x = \frac{2\sqrt{6}}{3}$$

$$f(0) = 0$$

$$f\left(-\frac{2\sqrt{6}}{3}\right) = \frac{16\sqrt{3}}{9}$$

$$f\left(\frac{2\sqrt{6}}{3}\right) = \frac{16\sqrt{3}}{9}$$

The relative minimum of the function is 0 at $x=0$.

The relative maximum of the function is $\frac{16\sqrt{3}}{9}$ at $x = -\frac{2\sqrt{6}}{3}$

The relative maximum of the function is $\frac{16\sqrt{3}}{9}$ at $x = \frac{2\sqrt{6}}{3}$

Thus $x=0, -\frac{2\sqrt{6}}{3}, \frac{2\sqrt{6}}{3}$

13. for concave up, the value of x is $-\frac{\sqrt{27-3\sqrt{33}}}{3}$

for concave down, x is: $x=2, x=\frac{\sqrt{27-3\sqrt{33}}}{3}$

At the point of inflection, x is $\frac{\sqrt{27-3\sqrt{33}}}{3}$

$$x = \frac{\sqrt{27-3\sqrt{33}}}{3}$$

$$14 \quad f(x) = x^2 \sqrt{4-x^2}$$

$$\Rightarrow f'(x) = \frac{d}{dx} (x^2 \sqrt{4-x^2})$$

$$\Rightarrow f'(x) = \frac{d}{dx} (x^2) \times \sqrt{4-x^2} + x^2 \times \frac{d}{dx} (\sqrt{4-x^2})$$

$$= f'(x) = 2x \sqrt{4-x^2} + x^2 \times \frac{1}{2\sqrt{4-x^2}} \times (-2x)$$

$$\Rightarrow f'(x) = \frac{8x - 3x^3}{\sqrt{4-x^2}}$$

$$\text{maximum value for } f = \frac{8\left(\frac{-2\sqrt{6}}{3}\right) - 3\left(\frac{-2\sqrt{6}}{3}\right)^3}{\sqrt{4 - \left(\frac{-2\sqrt{6}}{3}\right)^2}}$$

$$f = 0$$

$$\text{minimum value for } f = \frac{8(0) - 3(0)^3}{\sqrt{4 - (0)^2}}$$

$$f = 0$$

15

9) $f(x) = \frac{x^2 - 2x}{x+1}$

$$\lim_{x \rightarrow +\infty} \left(\frac{x^2 - 2x}{x+1} \right)$$

$$\lim_{x \rightarrow -\infty} \left(\frac{x^2 - 2x}{x+1} \right)$$

$\Rightarrow +\infty$
 $-\infty$

thus the function has no horizontal asymptotes

$$f(x) = \frac{x^2 - 2x}{x+1}$$

$$f(x) = \frac{x^2 - 2x}{x+1}, x \neq -1$$

$$\Rightarrow \lim_{x \rightarrow -1} \left(\frac{x^2 - 2x}{x+1} \right) \Rightarrow \text{does not exist}$$

thus the function has a vertical asymptote where $x = -1$

$$b) \quad g(x) = \frac{3x^2 - x + 5}{x^2 - 4}$$

$$\Rightarrow g(x) = \frac{3x^2 - x + 5}{x^2 - 4}, \quad x \neq -2, x \neq 2$$

$$\Rightarrow \lim_{x \rightarrow -2} \left(\frac{3x^2 - x + 5}{x^2 - 4} \right) \Rightarrow \text{does not exist}$$

$$\lim_{x \rightarrow 2} \left(\frac{3x^2 - x + 5}{x^2 - 4} \right) \Rightarrow \text{does not exist}$$

Thus the function has vertical asymptotes where $x = -2$
 $x = 2$

$$g(x) = \frac{3x^2 - x + 5}{x^2 - 4}$$

$$\Rightarrow \lim_{x \rightarrow +\infty} \left(\frac{3x^2 - x + 5}{x^2 - 4} \right) \Rightarrow 3$$

$$\lim_{x \rightarrow -\infty} \left(\frac{3x^2 - x + 5}{x^2 - 4} \right) \Rightarrow 3$$

The function has a horizontal asymptote where $y = 3$

$$10 \quad f'(x) = (t+1)^{-2} + e^{-2t}, \quad f(0) = 2$$

$$\Rightarrow f'(t) = \frac{d}{dt} \left((t+1)^{-2} + e^{-2t} \right)$$

$$\Rightarrow f'(t) = \frac{d}{dt} \left((t+1)^{-2} \right) + \frac{d}{dt} \left(e^{-2t} \right)$$

$$\Rightarrow f'(t) = -2(t+1)^{-3} + e^{-2t} \times (-2)$$

$$\Rightarrow \frac{(x(t^2 + 2t + e^{2t} + 1) + 2(t+1)^2 e^{2t}) e^{-2t}}{(t+1)^2}$$

12.

$$\begin{matrix} (31, 74) & , & (42, 89) \\ x_1, y_1 & & x_2, y_2 \end{matrix}$$

$$= \frac{89 - 74}{42 - 31} = \frac{15}{11}$$

Thus the net change is estimated to be $\frac{15}{11}$.

$$18$$

$$a) \int_1^3 (x^2 + \frac{1}{x}) dx$$

$$\Rightarrow \int x^2 + \frac{1}{x} dx$$

$$\Rightarrow \int x^2 dx + \int \frac{1}{x} dx$$

$$\Rightarrow \frac{x^3}{3} + \ln(|x|)$$

$$\Rightarrow \left(\frac{x^3}{3} + \ln(|x|) \right) \Big|_1^3$$

$$\Rightarrow \frac{3^3}{3} + \ln(3) - \left(\frac{1^3}{3} + \ln(1) \right)$$

$$\Rightarrow \frac{26}{3} + \ln(3)$$

$$\Rightarrow 9.76528$$

$$b) \int_0^{\pi/4} \sec^2(x) dx$$

$$\Rightarrow \int \sec^2(x) dx$$

$$\Rightarrow \tan(x)$$

$$\Rightarrow \tan(x) \Big|_0^{\pi/4}$$

$$\Rightarrow \tan(\pi/4) - \tan(0)$$

$$\Rightarrow 1$$

b).
19 Find dy/dx .

a)

$$y = \int_0^x \sqrt{1+t^3} dt$$

$$\Rightarrow y = \int_0^x \sqrt{1+t^3} dt$$

$$\Rightarrow y' = \frac{d}{dx} \left(\int_0^x \sqrt{1+t^3} dt \right)$$

$$\Rightarrow y' = \sqrt{1+x^3}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{1+x^3}$$

b) $y = \int_{2x}^{x^2} \sin t dt$

$$\Rightarrow y = \int_{2x}^0 \sin(t) dt + \int_0^{x^2} \sin(t) dt$$

$$\Rightarrow y' = \frac{d}{dx} \left(\int_{2x}^0 \sin(t) dt \right) + \frac{d}{dx} \left(\int_0^{x^2} \sin(t) dt \right)$$

$$\Rightarrow y' = -\sin(2x) \times 2 + \sin(x^2) \times 2x$$

$$\Rightarrow \frac{dy}{dx} = -2\sin(2x) + 2x \times \sin(x^2).$$

20 Evaluate : $\int \frac{\sin(x-3)}{\cos^3(x-3)} dx$

$\Rightarrow$ substituting $t = \cos(x-3)$

$\Rightarrow \int -\frac{1}{t^3} dt$

$\Rightarrow -\int \frac{1}{t^3} dt$

$\Rightarrow -\left(-\frac{1}{2t^2}\right)$

$\Rightarrow -\left(-\frac{1}{2\cos(x-3)^2}\right)$

$\Rightarrow \frac{1}{2\cos(x-3)^2}$

$\Rightarrow \frac{1}{2\cos(x-3)^2} + C$

$$21 \quad \int_e^{2e} \frac{dx}{x(\ln(x))^2}$$

$$\Rightarrow \int_e^{2e} \frac{1}{x(\ln(x))^2} dx$$

$$\Rightarrow \int \frac{1}{x \cdot \ln(x)^2} dx \quad \text{where } t = \ln(x).$$

$$\Rightarrow \int \frac{1}{t^2} dt$$

$$\Rightarrow -\frac{1}{t}$$

$$\Rightarrow -\frac{1}{\ln(x)}$$

$$\Rightarrow -\frac{1}{\ln(x)} \Big|_e^{2e}$$

$$\Rightarrow -\frac{1}{\ln(2e)} - \left(-\frac{1}{\ln(e)} \right)$$

$$\Rightarrow -\frac{1}{\ln(2)+1} + 1$$

$$\Rightarrow 0.409384$$